Tropical Intersections and Equilibrium

Elizabeth Baldwin Paul Klemperer

Nuffield College, Oxford University

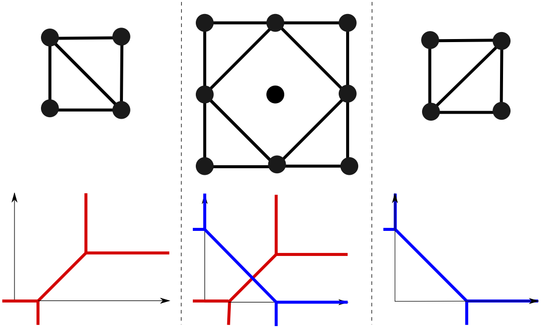
May 2016

Material from 'Tropical geometry to analyse demand' (2012-14) and 'Understanding preferences: "Demand types" and the existence of equilibrium with indivisibilities' (2015)

This work was supported by ESRC grant ES/L003058/1.

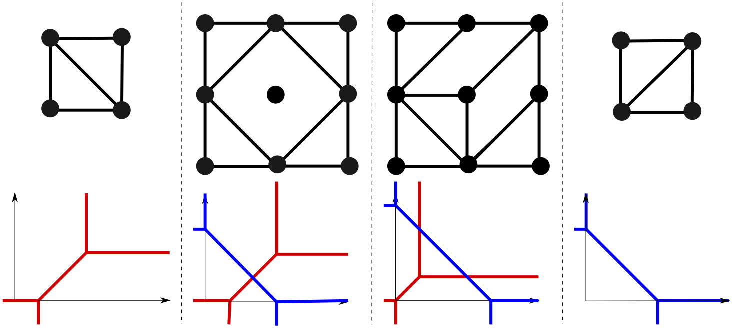
UNIVERSITY OF OXFORD

When unimodularity fails: 2-D Bézout-Bernstein



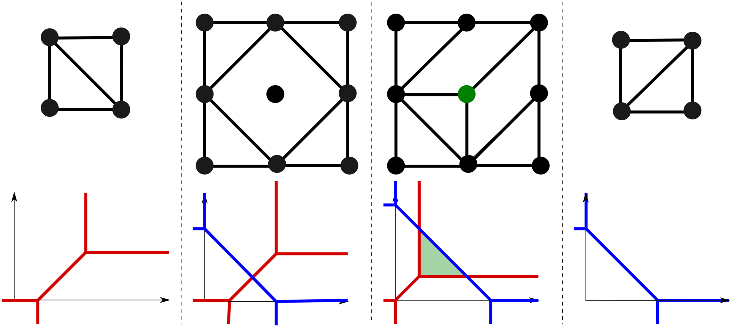
Return to substitutes / complements example.

When unimodularity fails: 2-D Bézout-Bernstein



Return to substitutes / complements example. Modify the valuations.

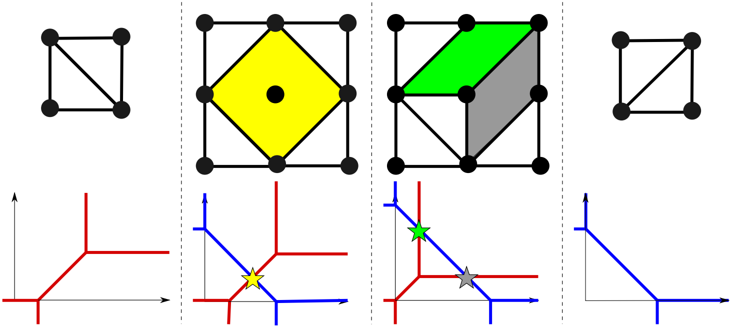
When unimodularity fails: 2-D Bézout-Bernstein



Return to substitutes / complements example. Modify the valuations. Now:

- Bundle $(1, 1)$ **is** demanded for some prices.
- Every bundle is demanded for some prices.

When unimodularity fails: 2-D Bézout-Bernstein

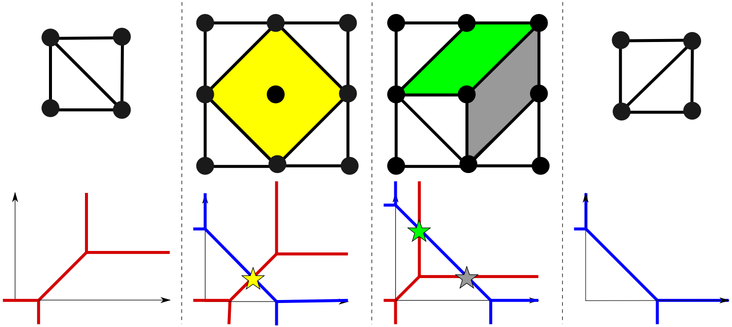


Before the shift

- One intersection.
- Corresp. demand complex cell has area 2.

After the shift

- Two intersections.
- Corresp. demand complex cells have area 1.



Before the shift

- One intersection.
- Corresp. demand complex cell has area 2.

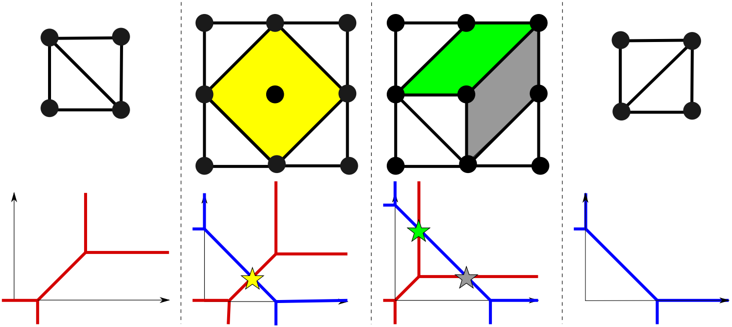
After the shift

- Two intersections.
- Corresp. demand complex cells have area 1.

Call this area the **multiplicity** of the intersection.

See # intersections is constant, **up to multiplicity**.

When unimodularity fails: 2-D Bézout-Bernstein



Definition

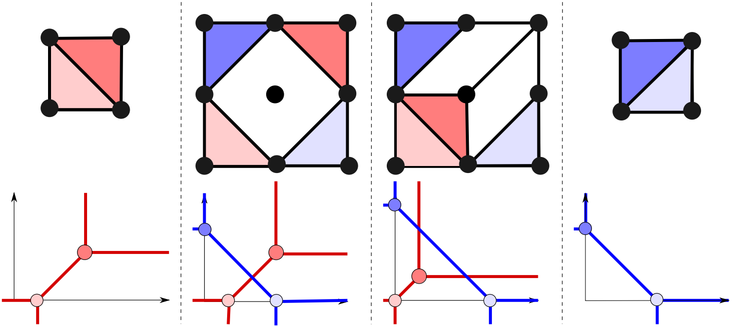
The 2-dimensional *mixed volume* is defined on convex sets X, Y:

$$MV_2(X,Y) := \mathrm{vol}_2(X+Y) - \mathrm{vol}_2(X) - \mathrm{vol}_2(Y).$$

Theorem (Tropical Bézout-Bernstein Theorem, see Sturmfels 2002)

$\#$ intersections, with multiplicities, is mixed volume of Newton Polytopes.

When unimodularity fails: 2-D Bézout-Bernstein



Definition

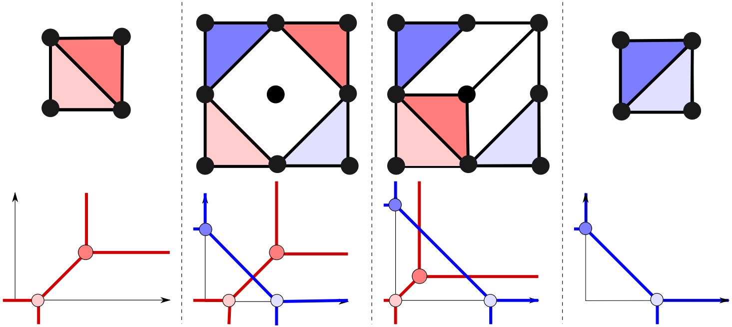
The 2-dimensional *mixed volume* is defined on convex sets X, Y:

$$MV_2(X,Y) := \mathrm{vol}_2(X+Y) - \mathrm{vol}_2(X) - \mathrm{vol}_2(Y).$$

Theorem (Tropical Bézout-Bernstein Theorem, see Sturmfels 2002)

$\#$ intersections, with multiplicities, is mixed volume of Newton Polytopes.

When unimodularity fails: 2-D Bézout-Bernstein



Theorem

When 2-D tropical hypersurfaces intersect transversally, then equilibrium exists for all relevant supply bundles iff # intersections, weighted only by facet weights, equals mixed volume of Newton Polytopes.

Proof. Equilibrium exists for all relevant supply bundles iff none of these cells contain additional non-vertex integer bundles.

Lattices and bases

Definition

- $N \subseteq \mathbb{Z}^n$ is a lattice if $\mathbf{n} - \mathbf{n}' \in N$ for all $\mathbf{n}, \mathbf{n}' \in N$.
- An (integer) basis for N is a set $\{\mathbf{n}^1, \ldots, \mathbf{n}^r\}$ such that any $\mathbf{n} \in N$ can be uniquely written $\mathbf{n} = \sum_i \alpha_i \mathbf{n}^i$ for $\alpha_i \in \mathbb{Z}$.

Facts

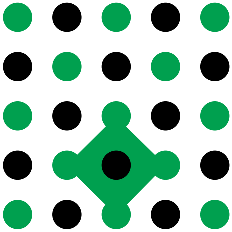
- If M, N are lattices then $M + N$ is a lattice.
- Every lattice has a basis.
- If the linear spans of M and N have zero intersection then the union of a basis for M and a basis for N is a basis for $M + N$.
- If N contains all the integer points in its linear span, i.e. $\langle N \rangle_{\mathbb{R}} \cap \mathbb{Z}^n = N$, then $\{\mathbf{n}^1, \ldots, \mathbf{n}^r\} \subset N$ is a basis iff it is linearly independent and unimodular.

Subgroup Index

- If N a lattice, then $N \cong \mathbb{Z}^k$ (use the basis to define an isomorphism).
- $\mathbb{Z}^k \subseteq \mathbb{R}^k$, so have vol_k for polytopes with vertices in $\mathbb{Z}^k$.
- Induce vol_N for polytopes with vertices in N.

Definition

- A fundamental parallelepiped Δ_N of N is $\{\sum_i \lambda_i \mathbf{n}^i \mid 0 \leq \lambda_i \leq 1\}$, where $\{\mathbf{n}^1, \ldots, \mathbf{n}^r\}$ are an basis for N.
- If $M \subseteq N$ then the subgroup Index $[N : M]$ is $\mathrm{vol}_N(\Delta_M)$.



Subgroup Index

- If N a lattice, then $N \cong \mathbb{Z}^k$ (use the basis to define an isomorphism).
- $\mathbb{Z}^k \subseteq \mathbb{R}^k$, so have vol_k for polytopes with vertices in $\mathbb{Z}^k$.
- Induce vol_N for polytopes with vertices in N.

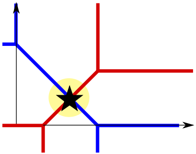
Definition

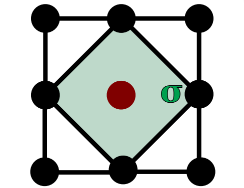
- A fundamental parallelepiped Δ_N of N is $\{\sum_i \lambda_i \mathbf{n}^i \mid 0 \leq \lambda_i \leq 1\}$, where $\{\mathbf{n}^1, \ldots, \mathbf{n}^r\}$ are an basis for N.
- If $M \subseteq N$ then the subgroup Index $[N : M]$ is $\mathrm{vol}_N(\Delta_M)$.

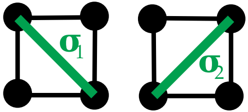
Facts

- $[N : M] = 1$ iff $N = M$
- If $\{\mathbf{n}^1, \ldots, \mathbf{n}^r\}$ is basis for $N \subseteq \mathbb{Z}^r$ then $[\mathbb{Z}^r : N] = |\det(\mathbf{n}^1, \ldots, \mathbf{n}^r)|$. So $\{\mathbf{n}^1, \ldots \mathbf{n}^r\}$ is unimodular iff $[\mathbb{Z}^r : N] = 1$
- There are $[N : M] - 1$ non-vertex points of N in Δ_M.

Alternate view on unimodularity theorem







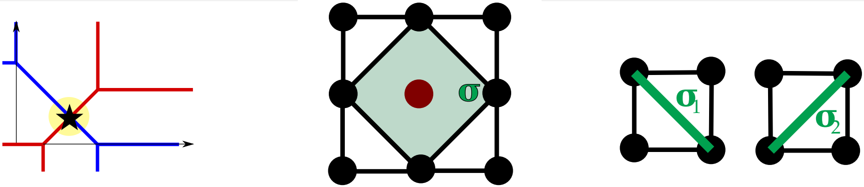
- 2 concave val'ns. Write $\sigma^i := \mathrm{Conv} D_{u_i}(\mathbf{p})$ and $\sigma := \mathrm{Conv} D_U(\mathbf{p})$.

Definition

- The parallel linear space $L_\sigma := \langle \{\mathbf{y} - \mathbf{x} : \mathbf{x}, \mathbf{y} \in \sigma\} \rangle_{\mathbb{R}}$ where σ is a demand complex cell.
- The parallel lattice $N_\sigma := \mathbb{Z}^n \cap L_\sigma$.

Lemma

$L_\sigma = L_{\sigma^1} \oplus L_{\sigma^2}$ *iff intersection is transverse at* $\mathbf{p}$



- 2 concave val'ns. Write $\sigma^i := \mathrm{Conv} D_{u_i}(\mathbf{p})$ and $\sigma := \mathrm{Conv} D_U(\mathbf{p})$.
- Let L_σ be linear space parallel to σ and $N_\sigma := \mathbb{Z}^n \cap L_\sigma$.
- Intersections transverse, so $L_\sigma = L_{\sigma^1} \oplus L_{\sigma^2}$.

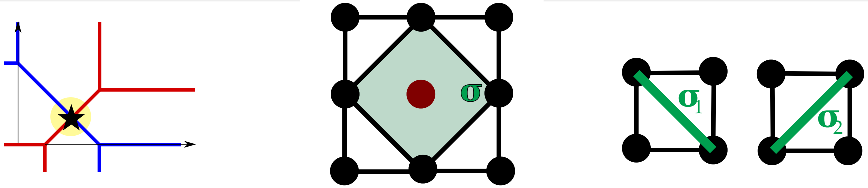
Definition

- The parallel linear space $L_\sigma := \langle \{\mathbf{y} - \mathbf{x} : \mathbf{x}, \mathbf{y} \in \sigma\} \rangle_{\mathbb{R}}$ where σ is a demand complex cell.
- The parallel lattice $N_\sigma := \mathbb{Z}^n \cap L_\sigma$.

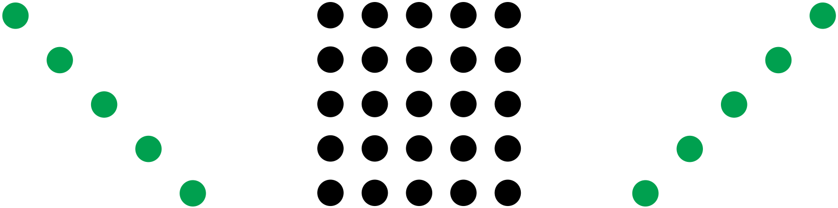
Lemma

$L_\sigma = L_{\sigma^1} \oplus L_{\sigma^2}$ *iff intersection is transverse at* $\mathbf{p}$

Alternate view on unimodularity theorem

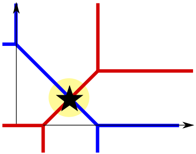


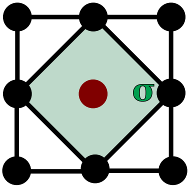
- 2 concave val'ns. Write $\sigma^i := \mathrm{Conv} D_{u_i}(\mathbf{p})$ and $\sigma := \mathrm{Conv} D_U(\mathbf{p})$.
- Let L_σ be linear space parallel to σ and $N_\sigma := \mathbb{Z}^n \cap L_\sigma$.
- Intersections transverse, so $L_\sigma = L_{\sigma^1} \oplus L_{\sigma^2}$.

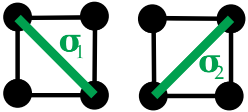


N_{σ^1} $\quad$ N_σ $\quad$ N_{σ^2}

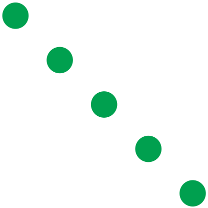
Alternate view on unimodularity theorem



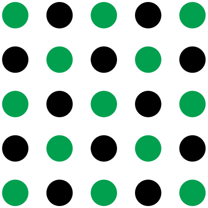




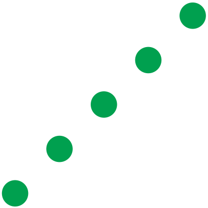
- 2 concave val'ns. Write $\sigma^i := \mathrm{Conv} D_{u_i}(\mathbf{p})$ and $\sigma := \mathrm{Conv} D_U(\mathbf{p})$.
- Let L_σ be linear space parallel to σ and $N_\sigma := \mathbb{Z}^n \cap L_\sigma$.
- Intersections transverse, so $L_\sigma = L_{\sigma^1} \oplus L_{\sigma^2}$.



N_{σ^1}

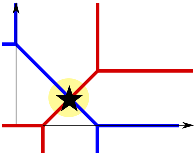


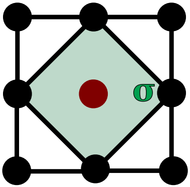
$[N_\sigma : N_{\sigma^1} + N_{\sigma^2}] = 2$

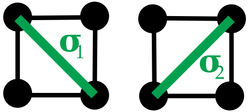


N_{σ^2}

Alternate view on unimodularity theorem





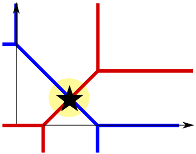


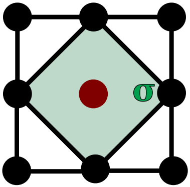
- 2 concave val'ns. Write $\sigma^i := \mathrm{Conv} D_{u_i}(\mathbf{p})$ and $\sigma := \mathrm{Conv} D_U(\mathbf{p})$.
- Let L_σ be linear space parallel to σ and $N_\sigma := \mathbb{Z}^n \cap L_\sigma$.
- Intersections transverse, so $L_\sigma = L_{\sigma^1} \oplus L_{\sigma^2}$.

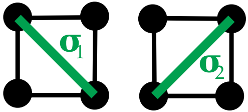
Proposition

If $\left[N_\sigma : N_{\sigma^1} + N_{\sigma^2}\right] = 1$ *then* $D_U(\mathbf{p})$ *is discrete-convex.*

Alternate view on unimodularity theorem





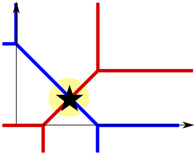


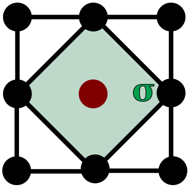
- 2 concave val'ns. Write $\sigma^i := \mathrm{Conv} D_{u_i}(\mathbf{p})$ and $\sigma := \mathrm{Conv} D_U(\mathbf{p})$.
- Let L_σ be linear space parallel to σ and $N_\sigma := \mathbb{Z}^n \cap L_\sigma$.
- Intersections transverse, so $L_\sigma = L_{\sigma^1} \oplus L_{\sigma^2}$.

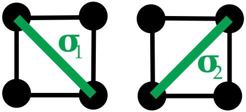
Proposition

If $\left[N_\sigma : N_{\sigma^1} + N_{\sigma^2}\right] = 1$ *then* $D_U(\mathbf{p})$ *is discrete-convex.*

Sketch proof. Let $\mathbf{x} \in \sigma \cap \mathbb{Z}^n$, so $\mathbf{x} = \mathbf{x}^1 + \mathbf{x}^2$ where $\mathbf{x}^i \in \sigma^i$. Is $\mathbf{x}^i \subseteq \mathbb{Z}^n$?







- 2 concave val'ns. Write $\sigma^i := \mathrm{Conv} D_{u_i}(\mathbf{p})$ and $\sigma := \mathrm{Conv} D_U(\mathbf{p})$.
- Let L_σ be linear space parallel to σ and $N_\sigma := \mathbb{Z}^n \cap L_\sigma$.
- Intersections transverse, so $L_\sigma = L_{\sigma^1} \oplus L_{\sigma^2}$.

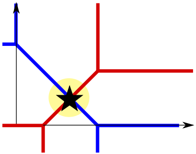
Proposition

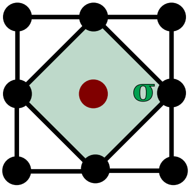
If $\left[N_\sigma : N_{\sigma^1} + N_{\sigma^2}\right] = 1$ *then* $D_U(\mathbf{p})$ *is discrete-convex.*

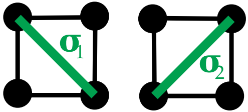
Sketch proof. Let $\mathbf{x} \in \sigma \cap \mathbb{Z}^n$, so $\mathbf{x} = \mathbf{x}^1 + \mathbf{x}^2$ where $\mathbf{x}^i \in \sigma^i$. Is $\mathbf{x}^i \subseteq \mathbb{Z}^n$?

Let $\mathbf{y} \in D_U(\mathbf{p})$, so $\mathbf{y} = \mathbf{y}^1 + \mathbf{y}^2$ where $\mathbf{y}^i \in D_{u^i}(\mathbf{p}) \in \mathbb{Z}^n$.

Alternate view on unimodularity theorem







- 2 concave val'ns. Write $\sigma^i := \mathrm{Conv} D_{u_i}(\mathbf{p})$ and $\sigma := \mathrm{Conv} D_U(\mathbf{p})$.
- Let L_σ be linear space parallel to σ and $N_\sigma := \mathbb{Z}^n \cap L_\sigma$.
- Intersections transverse, so $L_\sigma = L_{\sigma^1} \oplus L_{\sigma^2}$.

Proposition

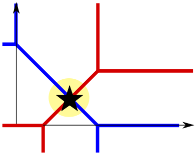
If $\left[N_\sigma : N_{\sigma^1} + N_{\sigma^2}\right] = 1$ *then* $D_U(\mathbf{p})$ *is discrete-convex.*

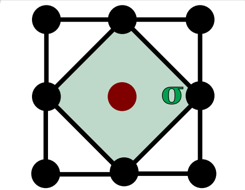
Sketch proof. Let $\mathbf{x} \in \sigma \cap \mathbb{Z}^n$, so $\mathbf{x} = \mathbf{x}^1 + \mathbf{x}^2$ where $\mathbf{x}^i \in \sigma^i$. Is $\mathbf{x}^i \subseteq \mathbb{Z}^n$?

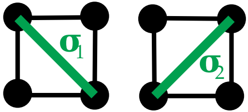
Let $\mathbf{y} \in D_U(\mathbf{p})$, so $\mathbf{y} = \mathbf{y}^1 + \mathbf{y}^2$ where $\mathbf{y}^i \in D_{u^i}(\mathbf{p}) \in \mathbb{Z}^n$.

And $\mathbf{y} - \mathbf{x} \in N_\sigma$ so $\mathbf{y} - \mathbf{x} = \mathbf{z}^1 + \mathbf{z}^2$, where $\mathbf{z}^i \in N_{\sigma^i} \subseteq \mathbb{Z}^n$.

Alternate view on unimodularity theorem







- 2 concave val'ns. Write $\sigma^i := \mathrm{Conv} D_{u_i}(\mathbf{p})$ and $\sigma := \mathrm{Conv} D_U(\mathbf{p})$.
- Let L_σ be linear space parallel to σ and $N_\sigma := \mathbb{Z}^n \cap L_\sigma$.
- Intersections transverse, so $L_\sigma = L_{\sigma^1} \oplus L_{\sigma^2}$.

Proposition

If $\left[N_\sigma : N_{\sigma^1} + N_{\sigma^2}\right] = 1$ *then* $D_U(\mathbf{p})$ *is discrete-convex.*

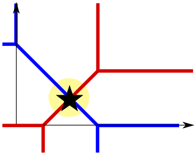
Sketch proof. Let $\mathbf{x} \in \sigma \cap \mathbb{Z}^n$, so $\mathbf{x} = \mathbf{x}^1 + \mathbf{x}^2$ where $\mathbf{x}^i \in \sigma^i$. Is $\mathbf{x}^i \subseteq \mathbb{Z}^n$?

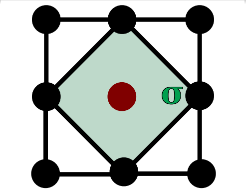
Let $\mathbf{y} \in D_U(\mathbf{p})$, so $\mathbf{y} = \mathbf{y}^1 + \mathbf{y}^2$ where $\mathbf{y}^i \in D_{u^i}(\mathbf{p}) \in \mathbb{Z}^n$.

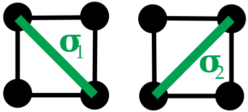
And $\mathbf{y} - \mathbf{x} \in N_\sigma$ so $\mathbf{y} - \mathbf{x} = \mathbf{z}^1 + \mathbf{z}^2$, where $\mathbf{z}^i \in N_{\sigma^i} \subseteq \mathbb{Z}^n$.

So $\mathbf{z}^1 + \mathbf{z}^2 = (\mathbf{y}^1 - \mathbf{x}^1) + (\mathbf{y}^2 - \mathbf{x}^2)$. By transversality $\mathbf{y}^i - \mathbf{x}^i = \mathbf{z}^i$.

Alternate view on unimodularity theorem







- 2 concave val'ns. Write $\sigma^i := \text{Conv} D_{u_i}(\mathbf{p})$ and $\sigma := \text{Conv} D_U(\mathbf{p})$.
- Let L_σ be linear space parallel to σ and $N_\sigma := \mathbb{Z}^n \cap L_\sigma$.
- Intersections transverse, so $L_\sigma = L_{\sigma^1} \oplus L_{\sigma^2}$.

Proposition

If $\left[N_\sigma : N_{\sigma^1} + N_{\sigma^2}\right] = 1$ *then* $D_U(\mathbf{p})$ *is discrete-convex.*

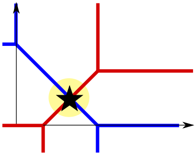
Sketch proof. Let $\mathbf{x} \in \sigma \cap \mathbb{Z}^n$, so $\mathbf{x} = \mathbf{x}^1 + \mathbf{x}^2$ where $\mathbf{x}^i \in \sigma^i$. Is $\mathbf{x}^i \subseteq \mathbb{Z}^n$?

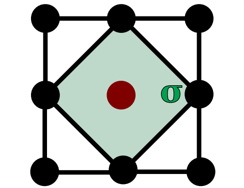
Let $\mathbf{y} \in D_U(\mathbf{p})$, so $\mathbf{y} = \mathbf{y}^1 + \mathbf{y}^2$ where $\mathbf{y}^i \in D_{u^i}(\mathbf{p}) \in \mathbb{Z}^n$.

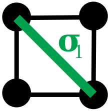
And $\mathbf{y} - \mathbf{x} \in N_\sigma$ so $\mathbf{y} - \mathbf{x} = \mathbf{z}^1 + \mathbf{z}^2$, where $\mathbf{z}^i \in N_{\sigma^i} \subseteq \mathbb{Z}^n$.

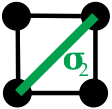
So $\mathbf{z}^1 + \mathbf{z}^2 = (\mathbf{y}^1 - \mathbf{x}^1) + (\mathbf{y}^2 - \mathbf{x}^2)$. By transversality $\mathbf{y}^i - \mathbf{x}^i = \mathbf{z}^i$.

Alternate view on unimodularity theorem





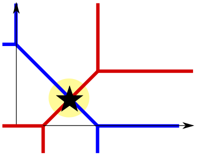


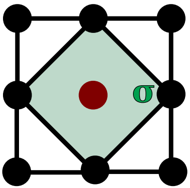


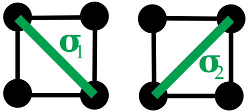
- m concave val'ns. Write $\sigma^i := \mathrm{Conv} D_{u_i}(\mathbf{p})$ and $\sigma := \mathrm{Conv} D_U(\mathbf{p})$.
- Let L_σ be linear space parallel to σ and $N_\sigma := \mathbb{Z}^n \cap L_\sigma$.
- Intersections transverse, so $L_\sigma = L_{\sigma^1} \oplus \cdots \oplus L_{\sigma^m}$.

Proposition

If $\left[N_\sigma : N_{\sigma^1} + \cdots + N_{\sigma^m}\right] > 1$ *and* $\dim \sigma^1 \leq 2$, $\dim \sigma^i = 1$ *for* $i = 2, .., m$ *then* $D_U(\mathbf{p})$ *not discrete-convex.*







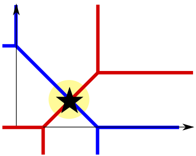
- m concave val'ns. Write $\sigma^i := \operatorname{Conv} D_{u_i}(\mathbf{p})$ and $\sigma := \operatorname{Conv} D_U(\mathbf{p})$.
- Let L_σ be linear space parallel to σ and $N_\sigma := \mathbb{Z}^n \cap L_\sigma$.
- Intersections transverse, so $L_\sigma = L_{\sigma^1} \oplus \cdots \oplus L_{\sigma^m}$.

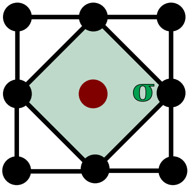
Proposition

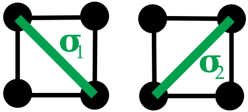
If $\left[N_\sigma : N_{\sigma^1} + \cdots + N_{\sigma^m}\right] > 1$ *and* $\dim \sigma^1 \leq 2$, $\dim \sigma^i = 1$ *for* $i = 2, .., m$ *then* $D_U(\mathbf{p})$ *not discrete-convex.*

Sketch proof. See $D_U(\mathbf{p}) = D_{u^1}(\mathbf{p}) + \cdots + D_{u^m}(\mathbf{p})$.
If $\mathbf{0} \in D_{u^j}(\mathbf{p})$ for all j then $D_U(\mathbf{p}) \subseteq N_{\sigma^1} + \cdots + N_{\sigma^m}$.
We know $\Delta_{N_{\sigma^1} + \cdots + N_{\sigma^m}}$ contains points not in $N_{\sigma^1} + \cdots + N_{\sigma^m}$.
Are such points in σ?

Alternate view on unimodularity theorem







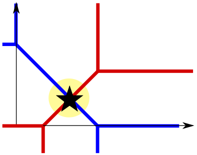
- m concave val'ns. Write $\sigma^i := \mathrm{Conv} D_{u_i}(\mathbf{p})$ and $\sigma := \mathrm{Conv} D_U(\mathbf{p})$.
- Let L_σ be linear space parallel to σ and $N_\sigma := \mathbb{Z}^n \cap L_\sigma$.
- Intersections transverse, so $L_\sigma = L_{\sigma^1} \oplus \cdots \oplus L_{\sigma^m}$.

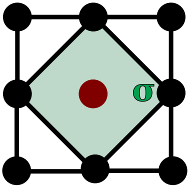
Proposition

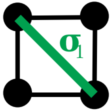
If $\left[N_\sigma : N_{\sigma^1} + \cdots + N_{\sigma^m}\right] > 1$ and $\dim \sigma^1 \leq 2$, $\dim \sigma^i = 1$ for $i = 2, .., m$ then $D_U(\mathbf{p})$ not discrete-convex.

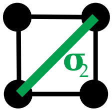
If $\dim \sigma^1 = 1$ then σ is itself (copies of a shift of) $\Delta_{N_{\sigma^1} + \cdots + N_{\sigma^m}}$.
So $D_U(\mathbf{p})$ is not discrete-convex.

Alternate view on unimodularity theorem









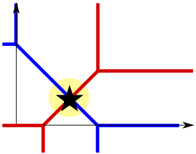
- m concave val'ns. Write $\sigma^i := \mathrm{Conv}\, D_{u_i}(\mathbf{p})$ and $\sigma := \mathrm{Conv}\, D_U(\mathbf{p})$.
- Let L_σ be linear space parallel to σ and $N_\sigma := \mathbb{Z}^n \cap L_\sigma$.
- Intersections transverse, so $L_\sigma = L_{\sigma^1} \oplus \cdots \oplus L_{\sigma^m}$.

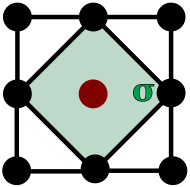
Proposition

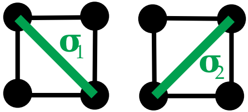
If $\left[N_\sigma : N_{\sigma^1} + \cdots + N_{\sigma^m}\right] > 1$ *and* $\dim \sigma^1 \leq 2$, $\dim \sigma^i = 1$ *for* $i = 2, .., m$ *then* $D_U(\mathbf{p})$ *not discrete-convex.*

Spse $\dim \sigma^1 = 2$.
Assume $\mathbf{0} \in \sigma^1$ & find a basis $\mathbf{n}^1, \mathbf{n}^2$ for N_{σ^1} such that $\mathbf{n}^1, \mathbf{n}^2 \in \sigma^1$.
Half of $\Delta_{N_{\sigma^1}}$ is in σ^1, so *half* of $\Delta_{N_{\sigma^1}+\cdots+N_{\sigma^m}}$ is in σ.
σ contains a point not in $N_{\sigma^1} + \cdots + N_{\sigma^m}$ by symmetry of $\Delta_{N_{\sigma^1}+\cdots+N_{\sigma^m}}$.

Alternate view on unimodularity theorem







- m concave val'ns. Write $\sigma^i := \mathrm{Conv} D_{u_i}(\mathbf{p})$ and $\sigma := \mathrm{Conv} D_U(\mathbf{p})$.
- Let L_σ be linear space parallel to σ and $N_\sigma := \mathbb{Z}^n \cap L_\sigma$.
- Intersections transverse, so $L_\sigma = L_{\sigma^1} \oplus \cdots \oplus L_{\sigma^m}$.

Proposition

If $\left[N_\sigma : N_{\sigma^1} + \cdots + N_{\sigma^m}\right] > 1$ *and* $\dim \sigma^1 \leq 2$, $\dim \sigma^i = 1$ *for* $i = 2, .., m$ *then* $D_U(\mathbf{p})$ *not discrete-convex.*

However, there exist examples in which $\left[N_\sigma : N_{\sigma^1} + N_{\sigma^2}\right] = 3$ and $\dim \sigma^1 = \dim \sigma^2 = 2$ but $D_U(\mathbf{p})$ *is* discrete-convex.

Example to show previous result is ‘tight’

Agent 1: $u^1(0,0,0,0) = 0$, $u^1(1,1,0,0) = 6$, $u^1(0,0,1,1) = 6$.
Agent 1 is indifferent between these bundles at

$$C_1 := \{\mathbf{p} \in \mathbb{R}^n : p_1 + p_2 = 6,\ p_3 + p_4 = 6.\}$$

Agent 2: $u^2(0,0,0,0) = 0$, $u^2(0,1,1,0) = 9$, $u^2(4,0,0,1) = 6$.
Agent 2 indifferent between these bundles at

$$C_2 := \{\mathbf{p} \in \mathbb{R}^n : p_2 + p_3 = 9,\ 4p_1 + p_4 = 6\}.$$

$C_1 \cap C_2 = (1,5,4,2) =: \mathbf{p}$. So

$$\sigma^1 = \mathrm{Conv}((0,0,0,0),(1,1,0,0),(0,0,1,1))$$
$$\sigma^2 = \mathrm{Conv}((0,0,0,0),(0,1,1,0),(4,0,0,1))$$
$$\sigma = \sigma^1 + \sigma^2$$

Example to show previous result is 'tight'

$$\sigma^1 = \text{Conv}((0,0,0,0),(1,1,0,0),(0,0,1,1))$$
$$\sigma^2 = \text{Conv}((0,0,0,0),(0,1,1,0),(4,0,0,1))$$
$$\sigma = \sigma^1 + \sigma^2$$

Example to show previous result is 'tight'

$$\sigma^1 = \mathrm{Conv}((0,0,0,0),(1,1,0,0),(0,0,1,1)) \qquad N_{\sigma^1} \text{ 2-dim'l}$$

$$\sigma^2 = \mathrm{Conv}((0,0,0,0),(0,1,1,0),(4,0,0,1)) \qquad N_{\sigma^2} \text{ 2-dim'l}$$

$$\sigma = \sigma^1 + \sigma^2 \qquad N_\sigma = \mathbb{Z}^4$$

Example to show previous result is 'tight'

$$\sigma^1 = \mathrm{Conv}((0,0,0,0),(1,1,0,0),(0,0,1,1)) \qquad N_{\sigma^1} \text{ 2-dim'l}$$

$$\sigma^2 = \mathrm{Conv}((0,0,0,0),(0,1,1,0),(4,0,0,1)) \qquad N_{\sigma^2} \text{ 2-dim'l}$$

$$\sigma = \sigma^1 + \sigma^2 \qquad N_\sigma = \mathbb{Z}^4$$

$\{(1,1,0,0),(0,0,1,1)\}$ are unimodular and so are a basis for N_{σ^1}.
$\{(0,1,1,0),(4,0,0,1)\}$ are unimodular and so are a basis for N_{σ^2}.

$$[N_\sigma : N_{\sigma^1} + N_{\sigma^2}] = \mathrm{abs}\det\begin{pmatrix} 1 & 0 & 0 & 4 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} = |1-4| = 3$$

Example to show previous result is 'tight'

$$\sigma^1 = \text{Conv}((0,0,0,0),(1,1,0,0),(0,0,1,1)) \qquad N_{\sigma^1} \text{ 2-dim'l}$$
$$\sigma^2 = \text{Conv}((0,0,0,0),(0,1,1,0),(4,0,0,1)) \qquad N_{\sigma^2} \text{ 2-dim'l}$$
$$\sigma = \sigma^1 + \sigma^2 \qquad N_\sigma = \mathbb{Z}^4$$

$\{(1,1,0,0),(0,0,1,1)\}$ are unimodular and so are a basis for N_{σ^1}.
$\{(0,1,1,0),(4,0,0,1)\}$ are unimodular and so are a basis for N_{σ^2}.
$[N_\sigma : N_{\sigma^1} + N_{\sigma^2}] = 3$

Example to show previous result is 'tight'

$$\sigma^1 = \mathrm{Conv}((0,0,0,0),(1,1,0,0),(0,0,1,1)) \qquad N_{\sigma^1} \text{ 2-dim'l}$$
$$\sigma^2 = \mathrm{Conv}((0,0,0,0),(0,1,1,0),(4,0,0,1)) \qquad N_{\sigma^2} \text{ 2-dim'l}$$
$$\sigma = \sigma^1 + \sigma^2 \qquad N_\sigma = \mathbb{Z}^4$$

$\{(1,1,0,0),(0,0,1,1)\}$ are unimodular and so are a basis for N_{σ^1}.
$\{(0,1,1,0),(4,0,0,1)\}$ are unimodular and so are a basis for N_{σ^2}.
$[N_\sigma : N_{\sigma^1} + N_{\sigma^2}] = 3$

$\Rightarrow$ two interior points in p'ped with these edges (containing σ).

Example to show previous result is 'tight'

$$\sigma^1 = \mathrm{Conv}((0,0,0,0),(1,1,0,0),(0,0,1,1)) \qquad N_{\sigma^1} \text{ 2-dim'l}$$
$$\sigma^2 = \mathrm{Conv}((0,0,0,0),(0,1,1,0),(4,0,0,1)) \qquad N_{\sigma^2} \text{ 2-dim'l}$$
$$\sigma = \sigma^1 + \sigma^2 \qquad N_\sigma = \mathbb{Z}^4$$

$\{(1,1,0,0),(0,0,1,1)\}$ are unimodular and so are a basis for N_{σ^1}.
$\{(0,1,1,0),(4,0,0,1)\}$ are unimodular and so are a basis for N_{σ^2}.
$[N_\sigma : N_{\sigma^1} + N_{\sigma^2}] = 3$

$\Rightarrow$ two interior points in p'ped with these edges (containing σ).

$$\frac{2}{3}\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + \frac{2}{3}\begin{pmatrix}0\\0\\1\\1\end{pmatrix} + \frac{1}{3}\begin{pmatrix}0\\1\\1\\0\end{pmatrix} + \frac{1}{3}\begin{pmatrix}4\\0\\0\\1\end{pmatrix} = \begin{pmatrix}2\\1\\1\\1\end{pmatrix}$$

Example to show previous result is 'tight'

$$\sigma^1 = \text{Conv}((0,0,0,0),(1,1,0,0),(0,0,1,1)) \qquad N_{\sigma^1} \text{ 2-dim'l}$$
$$\sigma^2 = \text{Conv}((0,0,0,0),(0,1,1,0),(4,0,0,1)) \qquad N_{\sigma^2} \text{ 2-dim'l}$$
$$\sigma = \sigma^1 + \sigma^2 \qquad N_\sigma = \mathbb{Z}^4$$

$\{(1,1,0,0),(0,0,1,1)\}$ are unimodular and so are a basis for N_{σ^1}.
$\{(0,1,1,0),(4,0,0,1)\}$ are unimodular and so are a basis for N_{σ^2}.
$[N_\sigma : N_{\sigma^1} + N_{\sigma^2}] = 3$

$\Rightarrow$ two interior points in p'ped with these edges (containing σ).

$$\frac{2}{3}\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + \frac{2}{3}\begin{pmatrix}0\\0\\1\\1\end{pmatrix} + \frac{1}{3}\begin{pmatrix}0\\1\\1\\0\end{pmatrix} + \frac{1}{3}\begin{pmatrix}4\\0\\0\\1\end{pmatrix} = \begin{pmatrix}2\\1\\1\\1\end{pmatrix} \notin \sigma^1 + \sigma^2$$

Example to show previous result is 'tight'

$$\sigma^1 = \mathrm{Conv}((0,0,0,0),(1,1,0,0),(0,0,1,1)) \qquad N_{\sigma^1} \text{ 2-dim'l}$$
$$\sigma^2 = \mathrm{Conv}((0,0,0,0),(0,1,1,0),(4,0,0,1)) \qquad N_{\sigma^2} \text{ 2-dim'l}$$
$$\sigma = \sigma^1 + \sigma^2 \qquad N_\sigma = \mathbb{Z}^4$$

$\{(1,1,0,0),(0,0,1,1)\}$ are unimodular and so are a basis for N_{σ^1}.
$\{(0,1,1,0),(4,0,0,1)\}$ are unimodular and so are a basis for N_{σ^2}.
$[N_\sigma : N_{\sigma^1} + N_{\sigma^2}] = 3$

$\Rightarrow$ two interior points in p'ped with these edges (containing σ).

$$\frac{1}{3}\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + \frac{1}{3}\begin{pmatrix}0\\0\\1\\1\end{pmatrix} + \frac{2}{3}\begin{pmatrix}0\\1\\1\\0\end{pmatrix} + \frac{2}{3}\begin{pmatrix}4\\0\\0\\1\end{pmatrix} = \begin{pmatrix}3\\1\\1\\1\end{pmatrix}$$

Example to show previous result is 'tight'

$$\sigma^1 = \mathrm{Conv}((0,0,0,0),(1,1,0,0),(0,0,1,1)) \qquad N_{\sigma^1} \text{ 2-dim'l}$$
$$\sigma^2 = \mathrm{Conv}((0,0,0,0),(0,1,1,0),(4,0,0,1)) \qquad N_{\sigma^2} \text{ 2-dim'l}$$
$$\sigma = \sigma^1 + \sigma^2 \qquad N_\sigma = \mathbb{Z}^4$$

$\{(1,1,0,0),(0,0,1,1)\}$ are unimodular and so are a basis for N_{σ^1}.
$\{(0,1,1,0),(4,0,0,1)\}$ are unimodular and so are a basis for N_{σ^2}.
$[N_\sigma : N_{\sigma^1} + N_{\sigma^2}] = 3$
$\Rightarrow$ two interior points in p'ped with these edges (containing σ).

$$\frac{1}{3}\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + \frac{1}{3}\begin{pmatrix}0\\0\\1\\1\end{pmatrix} + \frac{2}{3}\begin{pmatrix}0\\1\\1\\0\end{pmatrix} + \frac{2}{3}\begin{pmatrix}4\\0\\0\\1\end{pmatrix} = \begin{pmatrix}3\\1\\1\\1\end{pmatrix} \notin \sigma^1 + \sigma^2$$

Example to show previous result is 'tight'

$$\sigma^1 = \text{Conv}((0,0,0,0),(1,1,0,0),(0,0,1,1)) \qquad N_{\sigma^1} \text{ 2-dim'l}$$

$$\sigma^2 = \text{Conv}((0,0,0,0),(0,1,1,0),(4,0,0,1)) \qquad N_{\sigma^2} \text{ 2-dim'l}$$

$$\sigma = \sigma^1 + \sigma^2 \qquad N_\sigma = \mathbb{Z}^4$$

$\{(1,1,0,0),(0,0,1,1)\}$ are unimodular and so are a basis for N_{σ^1}.
$\{(0,1,1,0),(4,0,0,1)\}$ are unimodular and so are a basis for N_{σ^2}.
$[N_\sigma : N_{\sigma^1} + N_{\sigma^2}] = 3$

$\Rightarrow$ two interior points in p'ped with these edges (containing σ).

$$\frac{1}{3}\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + \frac{1}{3}\begin{pmatrix}0\\0\\1\\1\end{pmatrix} + \frac{2}{3}\begin{pmatrix}0\\1\\1\\0\end{pmatrix} + \frac{2}{3}\begin{pmatrix}4\\0\\0\\1\end{pmatrix} = \begin{pmatrix}3\\1\\1\\1\end{pmatrix} \notin \sigma^1 + \sigma^2$$

Neither of these points is in $\sigma = \text{Conv} D_U(\mathbf{p})$.
Every lattice point in σ is at a vertex. $D_U(\mathbf{p})$ is discrete-convex.

Equilibrium and intersections II

Recall

Lemma

Equilibrium fails for concave u^1, u^2 *and some supply* $\mathbf{x} \in A^1 + A^2$ *iff* $D_u(\mathbf{p})$ *is not discrete-convex for some* $\mathbf{p} \in \mathcal{T}_{u^1} \cap \mathcal{T}_{u^2}$.

Equilibrium and intersections II

Recall

Lemma

Equilibrium fails for concave u^1, u^2 and some supply $\mathbf{x} \in A^1 + A^2$ iff $D_u(\mathbf{p})$ is not discrete-convex for some $\mathbf{p} \in \mathcal{T}_{u^1} \cap \mathcal{T}_{u^2}$.

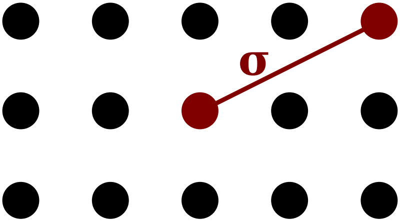
Not hard to show

Lemma

Equilibrium fails for concave u^1, u^2 and some supply $\mathbf{x} \in A^1 + A^2$ iff $D_u(\mathbf{p})$ is not discrete-convex for some $\mathbf{p}$ at a 0-cell of $\mathcal{T}_{u^1} \cap \mathcal{T}_{u^2}$.

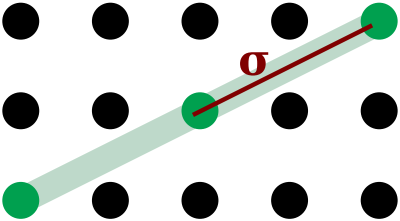
Cell Weights

Given $(n-2)$-cell C_σ of $\mathcal{T}_{u^1}$
$\longleftrightarrow$ k-cell σ of demand complex



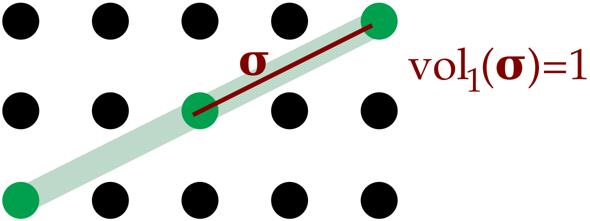
Cell Weights

Given $(n-2)$-cell C_σ of $\mathcal{T}_{u^1}$
$\longleftrightarrow$ k-cell σ of demand complex
Embed σ in N_σ.



Cell Weights

Given $(n-2)$-cell C_σ of $\mathcal{T}_{u^1}$
$\longleftrightarrow$ k-cell σ of demand complex
Embed σ in N_σ. Under $N_\sigma \cong \mathbb{Z}^k \hookrightarrow \mathbb{R}^k$ assign $\mathrm{vol}_{N_\sigma}(\sigma)$.

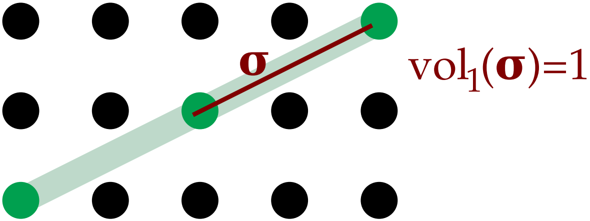


Cell Weights

Given $(n-2)$-cell C_σ of $\mathcal{T}_{u^1}$
$\longleftrightarrow$ k-cell σ of demand complex
Embed σ in N_σ. Under $N_\sigma \cong \mathbb{Z}^k \hookrightarrow \mathbb{R}^k$ assign $\mathrm{vol}_{N_\sigma}(\sigma)$.

Definition

The *weight* of cell C_σ is $w(C_\sigma) := k!\mathrm{vol}_k(\sigma)$.



Mixed Volume in General Dimension

Let $X_1, \ldots, X_n \subsetneq \mathbb{R}^n$ be convex sets.

Definition

The n-dimensional *mixed volume* of $X_1, \ldots, X_n \subset \mathbb{R}^n$ is

$$MV_n(X_1, \ldots, X_n) := \sum_{k=1}^{n} (-1)^{n-k} \left[\sum_{I \subset \{1,\ldots,n\} |I|=k} \mathrm{vol}_n \left(\sum_{i \in I} X_i \right) \right].$$

E.g.

$$\begin{aligned} MV_3(X, Y, Z) = {} & \mathrm{vol}_3(X + Y + Z) \\ & - \mathrm{vol}_3(X + Y) - \mathrm{vol}_3(Y + Z) - \mathrm{vol}_3(Z + X) \\ & + \mathrm{vol}_3(X) + \mathrm{vol}_3(Y) + \mathrm{vol}_3(Z) \end{aligned}$$

Mixed Volume in General Dimension

Let $X_1, \dots, X_n \subsetneq \mathbb{R}^n$ be convex sets.

Definition

The n-dimensional *mixed volume* of $X_1, \dots, X_n \subset \mathbb{R}^n$ is

$$MV_n(X_1, \dots, X_n) := \sum_{k=1}^{n} (-1)^{n-k} \left[\sum_{I \subset \{1,\dots,n\} |I|=k} \mathrm{vol}_n \left(\sum_{i \in I} X_i \right) \right].$$

E.g.

$$\begin{aligned} MV_3(X, Y, Z) = {} & \mathrm{vol}_3(X + Y + Z) \\ & - \mathrm{vol}_3(X + Y) - \mathrm{vol}_3(Y + Z) - \mathrm{vol}_3(Z + X) \\ & + \mathrm{vol}_3(X) + \mathrm{vol}_3(Y) + \mathrm{vol}_3(Z) \end{aligned}$$

Definition

$$MV_n(X, Y, (k, n-k)) := MV_n(X, \dots, X, Y, \dots, Y)$$

with k copies of X and $n - k$ copies of Y, for any $0 \leq k \leq n$.

Important examples of mixed volumes

In general, calculating mixed volumes is a problem with high complexity. But:

Fact (See e.g. Cox et al. 2005)

$$MV_n(X, \dots, X) = n!\mathrm{vol}_n(X)$$

Fact

If $A^j = \{\mathbf{x} \in \mathbb{Z}^n_{\geq 0} : \sum_i x_i \leq d_j\}$ *and* $X_j = \mathrm{Conv}\, A^j$, *for* $j = 1, 2$, *then*

$$MV_n(X_1, X_2, (n-k, k)) = d_1^{n-k} d_2^k$$

Generalised Bézout-Kouchnirenko-Bernstein

Write Δ_i for $\mathbb{R}$-convex hull of domain of valuation u^i, and $n_i := \dim \sigma^i$.

Theorem (see e.g. Bertrand and Bihan 2007, 2013)

If n_i-cells $C_{\sigma^1}, C_{\sigma^2}$ of $\mathcal{T}_{u^i}, \mathcal{T}_{u^2}$ intersect transversally at C_σ, then define the multiplicity of this intersection:

$$w(C_{\sigma^1})w(C_{\sigma^2})\left[N_\sigma : N_{\sigma^1} + N_{\sigma^2}\right] = MV(\sigma^1, \sigma^2, (n_1, n_2))$$

Using this multiplicity, the number of 0-cells in (transverse) $\mathcal{T}_{u^1} \cap \mathcal{T}_{u^2}$ is

$$\Gamma(\Delta_1, \Delta_2) := \sum_{k=1}^{n-1} MV(\Delta_1, \Delta_2, (k, n-k)).$$

Generalised Bézout-Kouchnirenko-Bernstein

Write Δ_i for $\mathbb{R}$-convex hull of domain of valuation u^i, and $n_i := \dim \sigma^i$.

Theorem (see e.g. Bertrand and Bihan 2007, 2013)

If n_i-cells $C_{\sigma^1}, C_{\sigma^2}$ of $\mathcal{T}_{u^i}, \mathcal{T}_{u^2}$ intersect transversally at C_σ, then define the multiplicity of this intersection:

$$w(C_{\sigma^1})w(C_{\sigma^2})\left[N_\sigma : N_{\sigma^1} + N_{\sigma^2}\right] = MV(\sigma^1, \sigma^2, (n_1, n_2))$$

Using this multiplicity, the number of 0-cells in (transverse) $\mathcal{T}_{u^1} \cap \mathcal{T}_{u^2}$ is

$$\Gamma(\Delta_1, \Delta_2) := \sum_{k=1}^{n-1} MV(\Delta_1, \Delta_2, (k, n-k)).$$

Recall:

- If $\left[N_\sigma : N_{\sigma^1} + N_{\sigma^2}\right] = 1$ every bundle in convex hull of aggregate demand at this price *is* demanded at this price.
- If $\left[N_\sigma : N_{\sigma^1} + N_{\sigma^2}\right] > 1$ and $n_1 \leq 2$, $n_2 = 1$ then there exists a relevant bundle, never demanded.

Generalised Bézout-Kouchnirenko-Bernstein

Write Δ_i for $\mathbb{R}$-convex hull of domain of valuation u^i, and $n_i := \dim \sigma^i$.

Theorem (see e.g. Bertrand and Bihan 2007, 2013)

If n_i-cells $C_{\sigma^1}, C_{\sigma^2}$ of $\mathcal{T}_{u^i}, \mathcal{T}_{u^2}$ intersect transversally at C_σ, then define the multiplicity of this intersection:

$$w(C_{\sigma^1})w(C_{\sigma^2})\left[N_\sigma : N_{\sigma^1} + N_{\sigma^2}\right] = MV(\sigma^1, \sigma^2, (n_1, n_2))$$

Using this multiplicity, the number of 0-cells in (transverse) $\mathcal{T}_{u^1} \cap \mathcal{T}_{u^2}$ is

$$\Gamma(\Delta_1, \Delta_2) := \sum_{k=1}^{n-1} MV(\Delta_1, \Delta_2, (k, n-k)).$$

Theorem

Suppose intersection $\mathcal{T}_{u^1} \cap \mathcal{T}_{u^2}$ is transverse.

1. *If count of 0-cells, weighted only by cell weights, equals $\Gamma(\Delta_1, \Delta_2)$, then equilibrium exists for every relevant supply.*

Generalised Bézout-Kouchnirenko-Bernstein

Write Δ_i for $\mathbb{R}$-convex hull of domain of valuation u^i, and $n_i := \dim \sigma^i$.

Theorem (see e.g. Bertrand and Bihan 2007, 2013)

If n_i-cells $C_{\sigma^1}, C_{\sigma^2}$ of $\mathcal{T}_{u^i}, \mathcal{T}_{u^2}$ intersect transversally at C_σ, then define the multiplicity of this intersection:

$$w(C_{\sigma^1})w(C_{\sigma^2})\left[N_\sigma : N_{\sigma^1} + N_{\sigma^2}\right] = MV(\sigma^1, \sigma^2, (n_1, n_2))$$

Using this multiplicity, the number of 0-cells in (transverse) $\mathcal{T}_{u^1} \cap \mathcal{T}_{u^2}$ is

$$\Gamma(\Delta_1, \Delta_2) := \sum_{k=1}^{n-1} MV(\Delta_1, \Delta_2, (k, n-k)).$$

Theorem

Suppose intersection $\mathcal{T}_{u^1} \cap \mathcal{T}_{u^2}$ is transverse.

1. *If count of 0-cells, weighted only by cell weights, equals $\Gamma(\Delta_1, \Delta_2)$, then equilibrium exists for every relevant supply.*
2. *If $n \leq 3$ and count of 0-cells, weighted by cell weights, strictly below $\Gamma(\Delta_1, \Delta_2)$ then equilibrium fails for some relevant supply.*

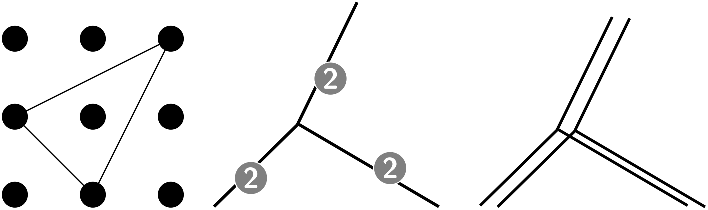
What if the intersection is not transverse?

We know that failure of equilibrium $\Rightarrow$ equilibrium still fails after a sufficiently small translation

The converse is *not* true

Example

Two identical agents with $D_{u^i}(\mathbf{p}) = \{(0,0),(1,2),(2,1),(1,1)\}$.Then $(2,2) = (1,1) + (1,1) \in D_U(\mathbf{p})$. But $(1,1)$ is demanded individually *only* at $\mathbf{p}$, so after a small translation of one agent, $(2,2) \in D_U(\mathbf{p})$ must fail.



Stable intersections

Recall the intersection $\mathcal{T}_{u^1} \cap \mathcal{T}_{u^2}$ is transverse at $\mathbf{p}$ if $\dim(C^1 + C^2) = n$, where C^j is the minimum cell of $\mathcal{T}_{u^j}$ containing $\mathbf{p}$.

Definition

The **stable intersection,** $\mathcal{T}_{u^1} \cap_{st} \mathcal{T}_{u^2}$, consists of all cell intersections $C^1 \cap C^2$ where C^i is a cell of $\mathcal{C}_{u^i}$ and where $\dim(C^1 + C^2) = n$.

So every transverse intersection price is stable.
Additional intersection prices may be stable.

Stable intersections

Recall the intersection $\mathcal{T}_{u^1} \cap \mathcal{T}_{u^2}$ is transverse at $\mathbf{p}$ if $\dim(C^1 + C^2) = n$, where C^j is the minimum cell of $\mathcal{T}_{u^j}$ containing $\mathbf{p}$.

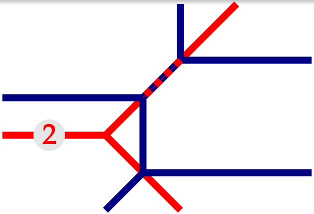
Definition

The **stable intersection,** $\mathcal{T}_{u^1} \cap_{st} \mathcal{T}_{u^2}$, consists of all cell intersections $C^1 \cap C^2$ where C^i is a cell of $\mathcal{C}_{u^i}$ and where $\dim(C^1 + C^2) = n$.

So every transverse intersection price is stable.
Additional intersection prices may be stable.

Proposition

For generic $\mathbf{w}$ *and small enough* $\epsilon > 0$,
$\mathcal{T}_{u^1} \cap_{st} \mathcal{T}_{u^2} = \lim_{\epsilon \to 0} \mathcal{T}_{u^1} \cap (\mathcal{T}_{u^2} + \epsilon \mathbf{w})$.



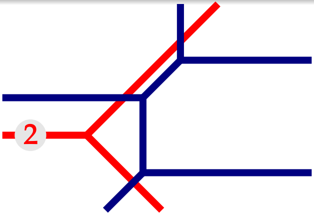
Stable intersections

Recall the intersection $\mathcal{T}_{u^1} \cap \mathcal{T}_{u^2}$ is transverse at $\mathbf{p}$ if $\dim(C^1 + C^2) = n$, where C^j is the minimum cell of $\mathcal{T}_{u^j}$ containing $\mathbf{p}$.

Definition

The **stable intersection,** $\mathcal{T}_{u^1} \cap_{st} \mathcal{T}_{u^2}$, consists of all cell intersections $C^1 \cap C^2$ where C^i is a cell of $\mathcal{C}_{u^i}$ and where $\dim(C^1 + C^2) = n$.

So every transverse intersection price is stable.
Additional intersection prices may be stable.



Proposition

For generic $\mathbf{w}$ *and small enough* $\epsilon > 0$,
$\mathcal{T}_{u^1} \cap_{st} \mathcal{T}_{u^2} = \lim_{\epsilon \to 0} \mathcal{T}_{u^1} \cap (\mathcal{T}_{u^2} + \epsilon \mathbf{w})$.

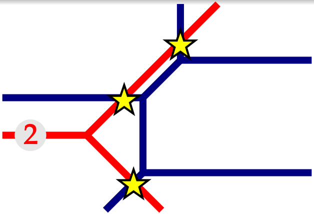
Stable intersections

Recall the intersection $\mathcal{T}_{u^1} \cap \mathcal{T}_{u^2}$ is transverse at $\mathbf{p}$ if $\dim(C^1 + C^2) = n$, where C^j is the minimum cell of $\mathcal{T}_{u^j}$ containing $\mathbf{p}$.

Definition

The **stable intersection,** $\mathcal{T}_{u^1} \cap_{st} \mathcal{T}_{u^2}$, consists of all cell intersections $C^1 \cap C^2$ where C^i is a cell of $\mathcal{C}_{u^i}$ and where $\dim(C^1 + C^2) = n$.

So every transverse intersection price is stable.
Additional intersection prices may be stable.



Proposition

For generic $\mathbf{w}$ and small enough $\epsilon > 0$,
$\mathcal{T}_{u^1} \cap_{st} \mathcal{T}_{u^2} = \lim_{\epsilon \to 0} \mathcal{T}_{u^1} \cap (\mathcal{T}_{u^2} + \epsilon \mathbf{w})$.

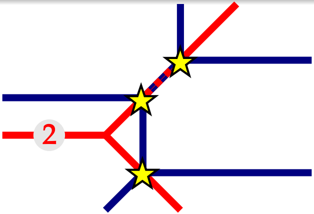
Stable intersections

Recall the intersection $\mathcal{T}_{u^1} \cap \mathcal{T}_{u^2}$ is transverse at $\mathbf{p}$ if $\dim(C^1 + C^2) = n$, where C^j is the minimum cell of $\mathcal{T}_{u^j}$ containing $\mathbf{p}$.

Definition

The **stable intersection,** $\mathcal{T}_{u^1} \cap_{st} \mathcal{T}_{u^2}$, consists of all cell intersections $C^1 \cap C^2$ where C^i is a cell of $\mathcal{C}_{u^i}$ and where $\dim(C^1 + C^2) = n$.

So every transverse intersection price is stable.
Additional intersection prices may be stable.



Proposition

For generic $\mathbf{w}$ *and small enough* $\epsilon > 0$,
$\mathcal{T}_{u^1} \cap_{st} \mathcal{T}_{u^2} = \lim_{\epsilon \to 0} \mathcal{T}_{u^1} \cap (\mathcal{T}_{u^2} + \epsilon \mathbf{w})$.

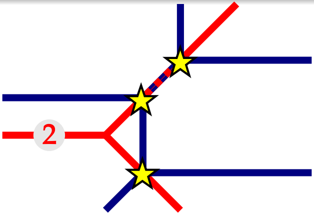
Stable intersections

Recall the intersection $\mathcal{T}_{u^1} \cap \mathcal{T}_{u^2}$ is transverse at $\mathbf{p}$ if $\dim(C^1 + C^2) = n$, where C^j is the minimum cell of $\mathcal{T}_{u^j}$ containing $\mathbf{p}$.

Definition

The **stable intersection,** $\mathcal{T}_{u^1} \cap_{st} \mathcal{T}_{u^2}$, consists of all cell intersections $C^1 \cap C^2$ where C^i is a cell of $\mathcal{C}_{u^i}$ and where $\dim(C^1 + C^2) = n$.

So every transverse intersection price is stable.
Additional intersection prices may be stable.



Proposition

For generic $\mathbf{w}$ *and small enough* $\epsilon > 0$,
$\mathcal{T}_{u^1} \cap_{st} \mathcal{T}_{u^2} = \lim_{\epsilon \to 0} \mathcal{T}_{u^1} \cap (\mathcal{T}_{u^2} + \epsilon \mathbf{w})$.

Lemma

Every 0*-cell in* $\mathcal{T}_{u^1} \cap \mathcal{T}_{u^2}$ *is in* $\mathcal{T}_{u^1} \cap_{st} \mathcal{T}_{u^2}$.

Using trop. Bézout-Bernstein for general intersections

Theorem

We may check for equilibrium by checking for failure of equilibrium at the 0*-cells of the stable intersection. Their number is bounded above by*

$$\Gamma(\Delta_1, \Delta_2) := \sum_{k=1}^{n-1} MV(\Delta_1, \Delta_2, (k, n-k)).$$

If this bound is tight then equilibrium exists for all relevant supplies.

Summary: Algorithm

Valuations u^1, u^2, aggregate U, domain $A \subset \mathbb{Z}^n$.

1. Are u^1, u^2 of the same unimodular demand type?

   Yes equilibrium exists $\forall \mathbf{x} \in A$.

2. If $\dim \mathrm{Conv}(A) = n' < n$, make basis change to $\mathbb{R}^{n'}$.
3. Number of 0-cells $\mathcal{T}_{u^1} \cap \mathcal{T}_{u^2}$ is $\sum_{k=1}^{n-1} MV_n(\tilde{A}_1, \tilde{A}_2, (n-k, k))$?

   Yes equilibrium exists $\forall \mathbf{x} \in A$.

   Yes, incl. cell wts; $\mathcal{T}_{u^1} \cap \mathcal{T}_{u^2}$ transverse equilibrium exists $\forall \mathbf{x} \in A$.

   No; $\mathcal{T}_{u^1} \cap \mathcal{T}_{u^2}$ transverse; $n \leq 3$ equilibrium fails for some $\mathbf{x} \in A$

4. For *each* k, does one of the following hold?

   (i)
   - $\mathcal{T}_{u^1} \cap \mathcal{T}_{u^2}$ transverse when a k-cell of $\mathcal{T}_{u^a}$ meets $(n-k)$-cell of $\mathcal{T}_{u^2}$.
   - Cell-weighted count of these 0-cells equals $MV_n(\tilde{A}_1, \tilde{A}_2, (n-k, k))$

   or (ii) for *every* 0-cell in both a k-cell of $\mathcal{T}_{u^1}$ and a $(n-k)$-cell of $\mathcal{T}_{u^2}$, the aggregate demand set is discrete-convex (found by other means).

   Yes equilibrium exists $\forall \mathbf{x} \in A$.

   No equilibrium fails for some $\mathbf{x} \in A$

Matching

We can apply our model to two forms of matching

- Coalition Formation (structure of possible coalitions)
- Bipartite Matching (preferences over matches)

See Baldwin and Klemperer (2012, 2014) and Baldwin and Klemperer (in preparation).

Coalition Formation

- Persons $i = 1, \ldots, n$; putative coalitions $\mathcal{X} \subseteq \{1, \ldots, n\}$.
- If coalition $\mathcal{X}$ forms, members gain 'intrinsic' value.
- Members can make side-payments – perfect transfers.

Coalition Formation

- Persons $i = 1, \ldots, n$; putative coalitions $\mathcal{X} \subseteq \{1, \ldots, n\}$.
- If coalition $\mathcal{X}$ forms, members gain 'intrinsic' value.
- Members can make side-payments – perfect transfers.

Total value of $\mathcal{X}$ is $u^{\mathcal{X}}(\mathcal{X})$. Shared among $i \in \mathcal{X}$ as 'utility payments'

$$\sum_{i \in \mathcal{X}} p_i \leq u^{\mathcal{X}}(\mathcal{X})$$

Coalition forms iff its value is no less than 'cost' of members.

Coalition Formation

- Persons $i = 1, \ldots, n$; putative coalitions $\mathcal{X} \subseteq \{1, \ldots, n\}$.
- If coalition $\mathcal{X}$ forms, members gain 'intrinsic' value.
- Members can make side-payments – perfect transfers.

Total value of $\mathcal{X}$ is $u^{\mathcal{X}}(\mathcal{X})$. Shared among $i \in \mathcal{X}$ as 'utility payments'

$$\sum_{i \in \mathcal{X}} p_i \leq u^{\mathcal{X}}(\mathcal{X})$$

Coalition forms iff its value is no less than 'cost' of members.

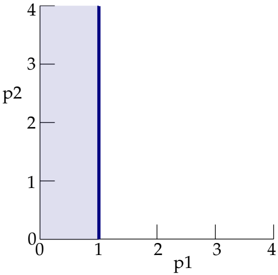
Key observation: Identical to an agent with valuation

$$u^{\mathcal{X}} : \{\mathbf{0}, \mathcal{X}\} \to \mathbb{R} \text{ s.t. } u(\mathbf{0}) = 0, u^{\mathcal{X}}(\mathcal{X}) \text{ as above.}$$

'Coalitions' are the agents. 'People' are the goods.

Coalition-agent examples

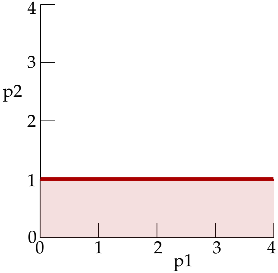
At what utilities (prices) do putative coalitions have potential to form? (The price for people not involved in the coalition does not matter).



- $A_{(1,0)}$, coalition $(1,0)$, with $u^{(1,0)}(1,0) = 1$

Coalition-agent examples

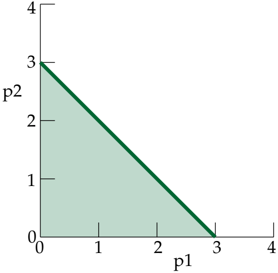
At what utilities (prices) do putative coalitions have potential to form? (The price for people not involved in the coalition does not matter).



- $A_{(0,1)}$, coalition $(0, 1)$ with $u^{(0,1)}(0, 1) = 1$

Coalition-agent examples

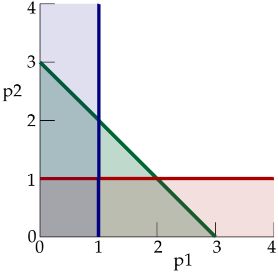
At what utilities (prices) do putative coalitions have potential to form? (The price for people not involved in the coalition does not matter).



- $A_{(1,1)}$, coalition $(1,1)$ with $u^{(1,1)}(1,1) = 3$

Coalition-agent examples

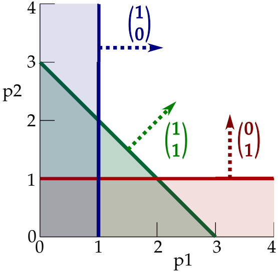
At what utilities (prices) do putative coalitions have potential to form? (The price for people not involved in the coalition does not matter).



- $A_{(1,0)}$, coalition $(1,0)$, with $u^{(1,0)}(1,0) = 1$
- $A_{(0,1)}$, coalition $(0,1)$ with $u^{(0,1)}(0,1) = 1$
- $A_{(1,1)}$, coalition $(1,1)$ with $u^{(1,1)}(1,1) = 3$

Coalition-agent examples

At what utilities (prices) do putative coalitions have potential to form? (The price for people not involved in the coalition does not matter).



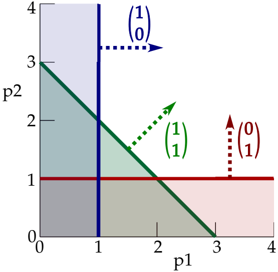
- $A_{(1,0)}$, coalition $(1, 0)$, with $u^{(1,0)}(1, 0) = 1$
- $A_{(0,1)}$, coalition $(0, 1)$ with $u^{(0,1)}(0, 1) = 1$
- $A_{(1,1)}$, coalition $(1, 1)$ with $u^{(1,1)}(1, 1) = 3$

Normal vector to price boundary is the coalition!

"Demand type" lists the putatitive coalitions.

Coalition-agent examples

At what utilities (prices) do putative coalitions have potential to form? (The price for people not involved in the coalition does not matter).



- $A_{(1,0)}$, coalition $(1,0)$, with $u^{(1,0)}(1,0) = 1$
- $A_{(0,1)}$, coalition $(0,1)$ with $u^{(0,1)}(0,1) = 1$
- $A_{(1,1)}$, coalition $(1,1)$ with $u^{(1,1)}(1,1) = 3$

Normal vector to price boundary is the coalition!

Response of 'people' to a choice of coalition?

- Choose the greatest net utility $(= u^i(\mathcal{X})$ + side-payments).
- So people 'behave like' goods being sold to the highest bidder

Suggests: stable matching $\Leftrightarrow$ existence of equilibrium prices.

Stability and equilibrium

Definition

A matching is **stable** if there is no 'blocking coalition' whose members would strictly prefer to leave their current match.

Theorem

A stable matching exists iff there exists a competitive equilibrium in the model with 'coalition-agents' and 'person-goods'.

Stability and equilibrium

Definition

A matching is **stable** if there is no 'blocking coalition' whose members would strictly prefer to leave their current match.

Theorem

A stable matching exists iff there exists a competitive equilibrium in the model with 'coalition-agents' and 'person-goods'.

Proof. Translate between matchings, and allocation with prices, easily:

- Each person (good) receives (costs) p_i in net utility.
- If coalition $\mathcal{X}$ forms then allocation to $A_{\mathcal{X}}$ is $\mathcal{X}$. Otherwise $\mathbf{0}$.

Stability and equilibrium

Definition

A matching is **stable** if there is no 'blocking coalition' whose members would strictly prefer to leave their current match.

Theorem

A stable matching exists iff there exists a competitive equilibrium in the model with 'coalition-agents' and 'person-goods'.

Proof. Translate between matchings, and allocation with prices, easily:

- Each person (good) receives (costs) p_i in net utility.
- If coalition $\mathcal{X}$ forms then allocation to $A_{\mathcal{X}}$ is $\mathcal{X}$. Otherwise $\mathbf{0}$.

1. Suppose allocation is competitive equilibrium at these prices.
   - If $A_{\mathcal{X}}$ receives $\mathbf{0}$, its value for $\mathcal{X}$ is too low relative to these prices.
     $\Rightarrow$ coalition $\mathcal{X}$ does not block the matching.

Stability and equilibrium

Definition

A matching is **stable** if there is no 'blocking coalition' whose members would strictly prefer to leave their current match.

Theorem

A stable matching exists iff there exists a competitive equilibrium in the model with 'coalition-agents' and 'person-goods'.

Proof. Translate between matchings, and allocation with prices, easily:

- Each person (good) receives (costs) p_i in net utility.
- If coalition $\mathcal{X}$ forms then allocation to $A_{\mathcal{X}}$ is $\mathcal{X}$. Otherwise $\mathbf{0}$.

2. Suppose matching is stable:
   - If coalition $\mathcal{X}$ forms, then $\sum_{i \in \mathcal{X}} p_i \leq u^{\mathcal{X}}(\mathcal{X})$
     $\Rightarrow$ agent $A_{\mathcal{X}}$ demands $\mathcal{X}$ at these prices.
   - If coalition $\mathcal{X}$ does not form, it does not block
     $\Rightarrow$ agent $A_{\mathcal{X}}$ happy to receive $\mathbf{0}$ at these prices.

Stability and equilibrium

Definition

A matching is **stable** if there is no 'blocking coalition' whose members would strictly prefer to leave their current match.

Theorem

A stable matching exists iff there exists a competitive equilibrium in the model with 'coalition-agents' and 'person-goods'.

Corollary

Fix a set of putative coalitions. A stable match exists for all possible valuations of these coalitions iff the set of vectors representing them satisfies both:

(a) *the supply* $(1, 1, \ldots, 1)$ *is in its convex hull;*

(b) **unimodularity**.

Examples when stable matching guaranteed

1. Workers and manager, as seen before.

Let $\mathcal{D}$ be the columns of:

$$\left(\begin{array}{ccccccccc} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right) \begin{array}{l} \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} \text{front-line workers} \\ \left.\begin{array}{c} \\ \end{array}\right\} \text{manager} \end{array}$$

Interpretation:

- The first three goods (rows) represent front-line workers.
- The final good (row) is a manager.
- 'Bundles', i.e. teams, worth bidding for, are:
    - a worker on their own (*not* a manager on their own);
    - a worker and a manager;
    - two workers and a manager.

Examples when stable matching guaranteed

1. Workers and manager, as seen before.
2. Two-sided matching: two types of person, coalitions are one of each.

$$
\begin{array}{r}
\text{'women'} \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. \\
\text{'men'} \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right.
\end{array}
\left(
\begin{array}{ccccccccc}
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
\hline
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1
\end{array}
\right)
$$

Examples when stable matching guaranteed

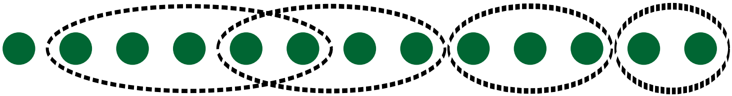
1. Workers and manager, as seen before.
2. Two-sided matching: two types of person, coalitions are one of each.

$$
\begin{array}{r}
\text{'women'} \\
\\
\text{'men'}
\end{array}
\begin{array}{l}
\left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. \\
\left\{ \begin{array}{c} \\ \\ \\ \end{array} \right.
\end{array}
\left. \begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
\hline
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix} \right\} \times -1: \text{ strong subs}
$$

Examples when stable matching guaranteed

1. Workers and manager, as seen before.
2. Two-sided matching: two types of person, coalitions are one of each.
3. Interval Packages: $\mathcal{D}$ consists of vectors $(0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)$

- People are ordered from 1 to n.
- Subsets of *consecutive* people can form coalitions.
- This is another basis change of strong substitutes.



This can represent firms, in a row, considering mergers (of any size):

- small shops along a street.
- owners of seabed rights for oil / offshore wind;

Bipartite Matching

- Separate agents into two sets, I and J.
- Seek to form many-to-one match. Consists of
  - One agent from I, 'firm'.
  - Many agents from J 'workers'.
- Agents have preferences over matches they can form.

Worker valuations are substitutes. If firm valuations are also substitutes, all of strong substitute demand type: stable matching guaranteed (Kelso and Crawford, 1982).

Bipartite Matching

- Separate agents into two sets, I and J.
- Seek to form many-to-one match. Consists of
  - One agent from I, 'firm'.
  - Many agents from J 'workers'.
- Agents have preferences over matches they can form.

- The goods are worker-firm pairs, (i, j).
- Each firm has valuation domain of form $\{0, 1\}^{|J|}$: can buy a relationship with any worker
- Each worker has valuation domain the *subset* of $\{-1, 0\}^{|I|}$ of vectors with at most one -1: can *sell* its labour to one firm.
- Stable matching: competitive equilibrium with supply $\mathbf{0}$.

Worker valuations are substitutes. If firm valuations are also substitutes, all of strong substitute demand type: stable matching guaranteed (Kelso and Crawford, 1982).

Summary

- Use of convex and tropical geometry helps in economics of indivisible goods, then applicable to auction design and matching.
  - Pretty pictures and intuition
  - Easier to generalise structures and results
- Some economics gives rise to interesting mathematical problems:
  - Consumer theory for indivisible goods
  - Mechanism design and auction theory
  - Matching (with transferable utility)
  - ...
- Thanks!

E. Baldwin and P. Klemperer. "Demand types", and stable multi-agent matching. In Preparation.

E. Baldwin and P. Klemperer. Tropical geometry to analyse demand. Mimeo., May 2012.

E. Baldwin and P. Klemperer. Tropical geometry to analyse demand. Mimeo. Available from www.paulklemperer.org and elizabeth-baldwin.me.uk, May 2014.

E. Baldwin and P. Klemperer. Understanding preferences: "demand types", and the existence of equilibrium with indivisibilities. Mimeo., February 2015.

B. Bertrand and F. Bihan. Euler characteristic of real nondegenerate tropical complete intersections. Technical Report 00380325, Hyper Articles en Ligne, 2007. e-print arXiv:0710.1222v2 [math.AG].

B. Bertrand and F. Bihan. Intersection multiplicity numbers between tropical hypersurfaces. In E. Brugalle, M. A. Cueto, A. Dickenstein, E.-M. Feichtner, and I. Itenberg, editors, *Algebraic and Combinatorial Aspects of Tropical Geometry*, volume 589. American Mathematical Soc., 2013.

D. A. Cox, J. B. Little, and D. O'Shea. *Using algebraic geometry*, volume 185 of *Graduate Texts in Mathematics*. Springer, 2005.

A. S. Kelso and V. P. Crawford. Job matching, coalition formation, and gross substitutes. *Econometrica*, 50(6):1483–1504, 1982.

B. Sturmfels. *Solving systems of polynomial equations*. Regional conference series in mathematics. Published for the Conference Board of the Mathematical Sciences by the American Mathematical Society, 2002.